

A Note on Absolute Derivations and Zeta Functions

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ABSTRACT

This note answers a question raised by Kurokawa, Ochiai and Wakayama, whether a certain operator constructed using a notion of quantum non-commutativity of primes has eigenvalues related to the Riemann zeta zeros.

1. Introduction

In studying the parallel between zeta functions of number fields and function fields over finite fields, certain properties of number fields seem describable by viewing them as geometric objects over the “field with one element.” Analogies in these directions have been formalized only recently, in Manin [5], Soulé [11], [12], Kurokawa, Ochiai and Wakayama [4], and Deitmar [2]. There is some earlier work, such as Kurokawa [3], which can be traced in the references in the papers above.

In particular, Kurokawa, Ochiai and Wakayama [4] recently introduced a notion of absolute derivation over the rational number field \mathbb{Q} . Based on this, they proposed a measure of “quantum non-commutativity” of pairs of primes over the rational field, given as follows. For real variables $x, y > 1$, define

$$F(x, y) = \sum_{k=1}^{\infty} x^{k-1} \frac{y^{-x^k}}{(1 - y^{-x^k})^2}. \quad (1.1)$$

Now define, for $x, y > 1$

$$QNC(x, y) := \frac{1}{12xy} (x(y-1)F(x, y) - y(x-1)F(y, x)). \quad (1.2)$$

The “quantum non-commutativity” of two primes p and q is defined to be $QNC(p, q)$. It is easy to see that $QNC(x, y) = -QNC(y, x)$, whence $QNC(x, x) = 0$, and one has $QNC(2, 3) = 0.00220482\dots$, for example. They then raised questions ([4, p. 580]) whether there is a connection between the quantum non-commutativity measure and zeta functions. Define the infinite skew-symmetric matrix $\mathbf{R} = [\mathbf{R}_{ij}]$ whose (i, j) -th entry

$$\mathbf{R}_{ij} := QNC(p_i, p_j),$$

where p_i denotes the i -th prime listed in increasing order, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ etc. In question (A) they asked whether it could be true (in some suitable sense) that

$$\det \left(\mathbf{I} - \mathbf{R}(s - \frac{1}{2}) \right) = c \xi(s), \quad (1.3)$$

in which $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, is the Riemann ξ -function and c is a nonzero constant. (They proposed $c = 2$.) They also asked a more general question (B) for (suitable) automorphic or Galois representations ρ , which would involve a skew-symmetric matrix $\mathbf{R}(\rho)$ with (i, j) -th entry

$$\mathbf{R}_{ij}(\rho) := \frac{\rho(p) + \rho(q)^*}{2} \mathbf{R}_{ij},$$

involving weighted version of elements $QNC(p_i, p_j)$, and asks whether it could be true that

$$\det \left(\mathbf{I} - \mathbf{R}(\rho)(s - \frac{1}{2}) \right) = c s^{m(\rho)} (s-1)^{m(\rho)} \hat{L}(s, \rho), \quad (1.4)$$

where $\hat{L}(s, \rho)$ is the completed L -function attached to the representation ρ , and $m(\rho)$ is the multiplicity of the trivial representation in ρ .

In order to make questions (A) and (B) well-defined one must formulate a suitable definition of infinite determinant in (1.3). We take as a basic requirement of such an infinite determinant that any zero s of a determinant (1.3) must necessarily have $z = \frac{1}{s-\frac{1}{2}}$ belonging to the spectrum of \mathbf{R} , i.e. that for this value the resolvent $(z\mathbf{I} - \mathbf{R})^{-1}$ is not a bounded operator on the full domain of \mathbf{R} , assumed to be a Banach space.

One consequence of this basic requirement is that if \mathbf{R} acts as a bounded operator on some Hilbert space in (1.3), then a positive answer to question (A) would necessarily imply the Riemann hypothesis. This follows since \mathbf{R} would then be skew-adjoint, hence have pure imaginary spectrum, whence the determinant (assumed defined) could only vanish when $s - \frac{1}{2}$ is pure imaginary. One can weaken question (A) so that it no longer implies the Riemann hypothesis, by requiring only that the left side $\det(\mathbf{I} - \mathbf{R}(s - \frac{1}{2}))$ of (1.3) detect all the zeta zeros that are on the critical line $\Re(s) = \frac{1}{2}$.

This note gives a negative answer to question (A) in both formulations. We treat the operator \mathbf{R} as acting on the Hilbert space l_2 of column vectors, and will show it is bounded. It follows that it is skew-adjoint and so has spectrum confined to the imaginary axis. However we show that its spectrum cannot detect ¹ all the zeta zeros that lie on the critical line, whether or not the Riemann hypothesis holds.

The main point is that the quantum non-commutativity function is so rapidly decreasing as p, q increase that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\mathbf{R}_{jk}| < \infty, \quad (1.5)$$

We show this in §2, and deduce that the matrix \mathbf{R} defines a trace class operator on l_2 . The weaker condition

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\mathbf{R}_{jk}|^2 < \infty, \quad (1.6)$$

¹If $\rho = \frac{1}{2} + i\gamma$ is a zeta zero, the corresponding point of the spectrum of \mathbf{R} is $\lambda = -\frac{i}{\gamma}$.

already implies that \mathbf{R} is a compact operator (in fact a Hilbert-Schmidt operator), see ² Akhiezer and Glazman [1, Sect. 28]. A compact operator necessarily has a pure discrete spectrum with all nonzero eigenvalues of finite multiplicity, with only limit point zero ([7, Theorem VI.15]. Since we now know \mathbf{R} is skew-adjoint, its eigenvalues, which necessarily occur in complex conjugate pure imaginary pairs, and can be ordered by decreasing absolute value, $\{\pm i\lambda_j : j = 1, 2, \dots\}$, with $\lambda_1 \geq \lambda_2 \geq \dots > 0$. A trace class operator \mathbf{A} is a compact operator with the property that its singular values μ_j (eigenvalues of the positive self-adjoint operator $(\mathbf{A}^* \mathbf{A})^{\frac{1}{2}}$) satisfy

$$\sum_{j=1}^{\infty} \mu_j < \infty. \quad (1.7)$$

For skew-adjoint operators $\mu_j = |\lambda_j|$, giving the condition

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty. \quad (1.8)$$

For trace class operators \mathbf{A} there is an essentially unique definition of $\det(I + \mathbf{A})$ that satisfies the basic requirement, see B. Simon [9], who reviews three equivalent definitions of this determinant (see also [10][Chap. 3]). He bases his treatment on the formula

$$\det(I - w\mathbf{A}) := \sum_{k=0}^{\infty} \text{Tr}(\bigwedge^k(w\mathbf{A})) = \sum_{k=0}^{\infty} \text{Tr}(\bigwedge^k \mathbf{A}) w^k.$$

which is also presented in Reed and Simon [8, Sec. XIII.17, p.323]. This determinant is an entire function in the variable w , given by the convergent infinite product

$$\det(I - w\mathbf{A}) = \prod_j (1 - w\lambda_j(\mathbf{A})),$$

which counts the eigenvalues $\lambda_j(\mathbf{A})$ of \mathbf{A} with their algebraic multiplicity, and which satisfies the basic requirement, Reed and Simon [8, Theorems XIII.105(c), XIII.106]. The truth of (1.3) for the trace class operator \mathbf{R} , taking $w = s - \frac{1}{2}$, would imply that if $s = \frac{1}{2} + i\gamma_j$ is a zeta zero on the critical line, then the two values $\lambda_j = \pm \frac{i}{\gamma_j}$ belong to the spectrum of \mathbf{R} . It is well known ([13, Chap. X]) that a positive proportion of zeta zeros lie on the critical line $\Re(s) = \frac{1}{2}$, and the asymptotics of these zeros easily give

$$\sum_{\{\gamma: \zeta(\frac{1}{2} + i\gamma) = 0\}} \frac{1}{|\gamma|} = +\infty. \quad (1.9)$$

This contradicts (1.8).

In §3 we discuss the problem of whether the notion of “QNC” can be modified to give a positive answer to question (A).

2. Proof

Our object is to show:

Theorem 2.1. *The operator \mathbf{R} acting on the column vector space l_2 defines a trace class operator.*

²In Akhiezer and Glazman, the term “completely continous operator” = “compact operator”.

Proof. A bounded operator \mathbf{A} is trace class if $|\mathbf{A}| = (\mathbf{A}^* \mathbf{A})^{\frac{1}{2}}$ is trace class, ie. the positive operator $|\mathbf{A}|$ has pure discrete spectrum and the sum of its eigenvalues converges cf. Reed and Simon [7, Sect VI.6]. A necessary and sufficient condition for an operator \mathbf{A} to be trace class is that for every orthonormal basis $\{\phi_n : 1 \leq n < \infty\}$ of l_2 one has

$$\sum_{n=1}^{\infty} |\langle \mathbf{A} \phi_n, \phi_n \rangle| < \infty \quad (2.1)$$

see Reed and Simon [7, Chapter VI, Ex. 26, p. 218].

Taking $\mathbf{A} = \mathbf{R}$, since it is skew-symmetric we have $\mathbf{R}^* \mathbf{R} = -\mathbf{R}^2$. It follows that if $|\mathbf{R}|$ is trace class, then it has pure discrete spectrum and the singular values of \mathbf{R} are just the absolute values of the eigenvalues of \mathbf{R} .

We now prove (1.5). We have

$$|QNC(p, q)| \leq \frac{1}{12} (F(p, q) + F(q, p))$$

Now we have $p, q \geq 2$ so $(1 - p^{-q^k})^2 \geq \frac{9}{16}$, whence

$$F(p, q) \leq \frac{16}{9} \sum_{k=1}^{\infty} p^{k-1} q^{-p^k} \leq 2q^{-p} + 2q^{-p} \left(\sum_{k=2}^{\infty} p^{k-1} q^{p-p^k} \right) \leq 6q^{-p}.$$

In the last step above we used ³ (for $k, p, q \geq 2$)

$$p^{k-1} q^{p-p^k} \leq p^{k-1} 2^{-p^{k-1}} \leq 2^{k-1} 2^{-2^{k-1}} \leq 2^{-k+2}.$$

This yields

$$|QNC(p, q)| \leq \frac{1}{2} (p^{-q} + q^{-p}),$$

from which we obtain

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\mathbf{R}_{jk}| \leq \sum_{m=2}^{\infty} \left(\sum_{n=m}^{\infty} m^{-n} \right) < \infty,$$

as asserted.

We use (1.5) to verify criterion (2.1). Let $\{e_k : 1 \leq k < \infty\}$ be the standard orthonormal basis of column vectore of l_2 , so that $\mathbf{R}(e_k) = \sum_{j=1}^{\infty} \mathbf{R}_{jk} e_j$. Now let $\phi_n = \sum_{k=1}^{\infty} c_{nk} e_k$ be an orthonormal basis of l_2 , so that $[c_{nk}]$ is a unitary matrix. Then we have $\|\phi_n\|^2 = \sum_{k=1}^{\infty} |c_{nk}|^2 = 1$, and unitarity also implies

$$\sum_{n=1}^{\infty} |c_{nk}|^2 = 1. \quad (2.2)$$

Now we compute

$$\begin{aligned} \sum_{n=1}^{\infty} |\langle \mathbf{R} \phi_n, \phi_n \rangle| &= \sum_{n=1}^{\infty} \left| \left\langle \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{nk} \mathbf{R}_{jk} e_j, \sum_{j=1}^{\infty} c_{nj} e_j \right\rangle \right| \\ &\leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |c_{nk} \mathbf{R}_{jk} \overline{c_{nj}}| \end{aligned}$$

³Note that $x2^{-x}$ is decreasing for $x \geq 2 > \frac{1}{\log 2}$.

$$\begin{aligned}
&\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\mathbf{R}_{jk}| \left(\sum_{n=1}^{\infty} |c_{nj}| |c_{nk}| \right) \\
&\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\mathbf{R}_{jk}| \left(\sum_{n=1}^{\infty} \frac{1}{2} (|c_{nj}|^2 + |c_{nk}|^2) \right) \\
&\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\mathbf{R}_{jk}| < \infty
\end{aligned}$$

as required. ■

3. Concluding Remarks

It is an interesting question whether the concept of “QNC” has a natural modification to correct the difficulty observed here, and possibly to give a positive answer to question (A). We have no proposal how to do this, but make the following remarks.

The argument made above rests on the following fact: A necessary condition on a skew-symmetric compact operator \mathbf{R} acting on a Hilbert space to have a determinant (1.3) satisfying the basic requirement that detects the zeta function zeros is that it be a Hilbert-Schmidt operator which is not of trace class. In order to define (1.3) when \mathbf{R} is a Hilbert-Schmidt operator that is not of trace class, an extended definition of infinite determinant is required. There are notions of regularized determinant $\det(I + w\mathbf{A})$ that apply to Hilbert-Schmidt operators \mathbf{A} and satisfy the basic requirement. One such, denoted $\det_2(I + w\mathbf{A})$, is discussed in Simon [9] and Simon [10, Chap. 3]. See Pietsch [6, Chapters 4, 7] for further work on such questions.

The results of Kurokawa, Ochiai and Wakayama [4] were motivated in part by the function field case for the absolute function field $K = \mathbb{F}_q(T)$, as noted at the beginning of their paper. We note that one might reconsider the function field analogy, varying the base function field. For the (absolute) function field case $\mathbb{F}_q(T)$ the corresponding matrix (and operator) $\mathbf{R} \equiv 0$, but if one allowed other function fields K of genus one or higher, then the function field analogue of the quantity (1.9) also diverges. This holds because the function field zeta zeros $\frac{1}{2} + i\gamma$ have γ falling in a finite number of arithmetic progressions (mod $\frac{2\pi}{\log p}$), so that

$$\sum_{\gamma} \frac{1}{|\gamma|} = +\infty.$$

Thus the difficulty above manifests itself already in the function field case. It therefore might be useful to look for formulas for quantum non-commutativity for prime ideals in a function field K of genus at least one, intending to construct an analogous matrix \mathbf{R}_K . The operator corresponding to \mathbf{R}_K on l_2 would necessarily be Hilbert-Schmidt, but not of trace class, if it were to have eigenvalues $\pm \frac{i}{\gamma}$, where $\frac{1}{2} + i\gamma$ runs over the function field zeta zeros of K , counted with multiplicity. Perhaps such study could clarify the notion of “QNC”.

Finally we note that if to the sum defining the function $F(x, y)$ in (1.1) the term $k = 0$ were added, the definition of $QNC(p, q)$ would be modified to add the extra terms

$$\frac{1}{12pq} \left(\frac{1}{q-1} - \frac{1}{p-1} \right).$$

The resulting modified operator $\tilde{\mathbf{R}}$ then has

$$\sum_{i,j} |\tilde{\mathbf{R}}_{ij}| = +\infty,$$

and is a Hilbert-Schmidt operator on l_2 not of trace class.

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